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# On the Transformation of a Stationary Fuzzy Random Process by a Linear Dynamic System

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**Abstract**—In this paper, stationary random processes with fuzzy states are studied. The properties of their numerical characteristics—fuzzy expectations, expectations, and covariance functions—are established. The spectral representation of the covariance function, the generalized Wiener–Khinchin theorem, is proved. The main attention is paid to the problem of transforming a stationary fuzzy random process (signal) by a linear dynamic system. Explicit-form relationships are obtained for the fuzzy expectations (and expectations) of input and output stationary fuzzy random processes. An algorithm is developed and justified to calculate the covariance function of a stationary fuzzy random process at the output of a linear dynamic system from the covariance function of a stationary input fuzzy random process. The results rest on the properties of fuzzy random variables and numerical random processes. Triangular fuzzy random processes are considered as examples.

*Keywords*: stationary random processes, fuzzy states, fuzzy expectations, covariance functions, the transformation of a fuzzy random process by a linear dynamic system

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## 1. INTRODUCTION

In this paper, continuous random processes with fuzzy states (fuzzy random processes) are studied. Specifically, the time variable and the set of possible fuzzy states are considered to be continuous. The cross-section of a continuous fuzzy random process at any time instant is a fuzzy random variable. We apply the well-known results of fuzzy modeling [1, 2] and the theory of fuzzy random variables [3–5] and classical results of the theory of real random processes [6, 7].

This work continues the previous research on the theory of continuous random processes with fuzzy states [8]. In particular, the properties of the fuzzy expectations, expectations, and covariance functions of continuous fuzzy random processes were studied, and the relationship between the characteristics of fuzzy random signals at the input and output of a linear dynamic system was investigated using the Green function method.

Below, we introduce and analyze stationary fuzzy random processes, proving the spectral representation of the covariance function (the generalized Wiener–Khinchin theorem). Based on this theorem, we propose and justify an algorithm for calculating the characteristics of a stationary fuzzy random process (signal) at the output of a linear dynamic system, namely, its fuzzy expectation, expectation, and covariance functions, from the corresponding characteristics of the input fuzzy random process (signal). The results obtained in this area develop to the fuzzy case the well-known ones for real continuous random processes; for example, see [6, Chapter 7; 7, Chapter VII].

Let us emphasize the difference between the approach and results of this paper and the studies on continuous-time random processes with discrete fuzzy states. For example, fuzzy queueing systems were discussed in [9–12]; stochastic fuzzy dynamic automatic control systems were considered in [13, 14]. At the same time, stationary fuzzy random processes and their covariance functions were not addressed in the publications cited above.

In what follows, a fuzzy number $\tilde{z}$ defined on the universal space $R$ of real numbers is understood as a set of ordered pairs $(x, \mu_{\tilde{z}}(x))$, where the membership function $\mu_{\tilde{z}} : R \to [0, 1]$ determines the degree (grade) of membership $\forall\, x \in R$ to the set $\tilde{z}$ [1, Chapter 5]. In this paper, the interval representation of fuzzy numbers is used [1, Chapter 5]. In this case, the $\alpha$-level set of a fuzzy number $\tilde{z}$ with a membership function $\mu_{\tilde{z}}(x)$ is defined as $Z_\alpha = \{x | \mu_{\tilde{z}}(x) \geqslant \alpha\}$ $(\alpha \in (0, 1])$, $Z_0 = cl\{x|\mu_{\tilde{z}}(x) > 0\}$, where $cl$ indicates the closure of an appropriate set.

Assume that all $\alpha$-levels of a fuzzy number are closed and bounded intervals of the real axis. Thus, $Z_\alpha = [z^-(\alpha), z^+(\alpha)]$, where $z^-(\alpha)$ and $z^+(\alpha)$ are the left and right $\alpha$-indices of a fuzzy number, respectively.

We will consider the set $J$ of fuzzy numbers for which the indices $z^\pm(\alpha)$ satisfy the following standard conditions:

1. $z^-(\alpha) \leqslant z^+(\alpha)$, $\forall\, \alpha \in [0, 1]$.

2. The function $z^-(\alpha)$ is bounded, nondecreasing, left-continuous on the interval $(0, 1]$, and right-continuous at the point 0.

3. The function $z^+(\alpha)$ is bounded, nonincreasing, left-continuous on the interval $(0, 1]$, and right-continuous at the point 0.

The sum of fuzzy numbers is a fuzzy number whose indices represent the sums of the corresponding indices of the summands. Multiplication of a fuzzy number by a positive real number means multiplying its indices by the latter number. Multiplication of a fuzzy number by a negative real number means multiplying its indices by the latter number and reversing them. Equality of fuzzy numbers is understood as equality of the corresponding $\alpha$-indices ($\forall\, \alpha \in [0, 1]$).

A real number $r$ is associated with a fuzzy number whose left and right $\alpha$-indices coincide with $r$ $\forall\, \alpha \in [0, 1]$.

## 2. THE FUZZY EXPECTATIONS, EXPECTATIONS, AND COVARIANCES OF FUZZY RANDOM VARIABLES

Let $(\Omega, \Sigma, P)$ be a probability space, where $\Omega$ denotes the set of elementary events, $\Sigma$ is a $\sigma$-algebra consisting of all subsets of the set $\Omega$, and $P$ is a probability measure. Consider a mapping $\tilde{X} : \Omega \to J$. For a fixed $\omega \in \Omega$, its $\alpha$-level intervals $X_\alpha(\omega)$ are given by $X_\alpha(\omega) = \{r \in R : \mu_{\tilde{X}(\omega)}(r) \geqslant \alpha\}$ $\alpha \in (0, 1]$, $X_0(\omega) = cl\{\mu_{\tilde{X}(\omega)}(r) > 0\}$, where $\mu_{\tilde{X}(\omega)}(r)$ means the membership function of the fuzzy number $\tilde{X}(\omega)$. An interval $X_\alpha(\omega)$ can be represented as $X_\alpha(\omega) = [X^-(\omega, \alpha), X^+(\omega, \alpha)]$, and the bounds $X^-(\omega, \alpha)$ and $X^+(\omega, \alpha)$ are called the left and right $\alpha$-indices of $\tilde{X}(\omega)$, respectively.

A mapping $\tilde{X} : \Omega \to J$ is called a fuzzy random variable (FRV) if the real-valued functions $X^\pm(\omega, \alpha)$ are measurable in $\omega$ $\forall\, \alpha \in [0, 1]$; for example, see [3, 4]. In this case, $\alpha$-indices are real random variables $\forall\, \alpha \in [0, 1]$.

We will consider the class $\mathscr{X}$ of FRVs for which the indices $X^-(\omega, \alpha)$ and $X^+(\omega, \alpha)$ are square summable on $\Omega \times [0, 1]$. Let

$$x^-(\alpha) = EX^-(\omega, \alpha), \qquad x^+(\alpha) = EX^+(\omega, \alpha). \tag{1}$$

Throughout this paper, $E$ means the expectation of a random variable, i.e., $E\xi = \int_\Omega \xi(\omega)\, dP$ for a random variable $\xi(\omega)$.

A fuzzy number with the indices given by (1) is called the fuzzy expectation of an FRV $\tilde{X}$ and denoted by $M(\tilde{X})$; its indices are denoted by $[M(\tilde{X})]^{\pm}_{\alpha}$.

The expectation $m(\tilde{X})$ of an FRV $X \in \mathcal{X}$ is defined as the mean of a fuzzy number $M(\tilde{X})$ with the $\alpha$-indices $M^{\pm}(\alpha)$ given by (1):

$$m(\tilde{X}) = \frac{1}{2}\int_0^1 \Big([M(\tilde{X})]^-(\alpha) + [M(\tilde{X})]^+(\alpha)\Big)\, d\alpha. \tag{2}$$

(For details, see [15].)

The covariance of FRVs $\tilde{X}$ and $\tilde{Y}$ is defined as

$$cov(\tilde{X}, \tilde{Y}) = \frac{1}{2}\int_0^1 \left(cov(X^-_{\alpha}, Y^-_{\alpha}) + cov(X^+_{\alpha}, Y^+_{\alpha})\right) d\alpha, \tag{3}$$

and the variance of an FRV $\tilde{X}$ is defined as $D(\tilde{X}) = cov(\tilde{X}, \tilde{X})$ [4]. In the expression (3), the covariances of real random variables $X^{\pm}_{\alpha}$ and $Y^{\pm}_{\alpha}$ are given by the conventional formula $cov(X^{\pm}_{\alpha}, Y^{\pm}_{\alpha})E(X^{\pm}_{\alpha} - E(X^{\pm}_{\alpha}))(Y^{\pm}_{\alpha} - E(Y^{\pm}_{\alpha}))$ [16, Chapter 14].

The properties of the fuzzy expectations, expectations, covariances, and variances of FRVs were discussed in [4, 5, 17, Chapter 6].

## 3. CONTINUOUS RANDOM PROCESSES WITH FUZZY STATES

In Sections 3 and 4, we will use the notion of the limit, continuity, and differentiability of real random processes in the mean-square (m.s.) sense. Consider the Hilbert space $\mathcal{H}$ of real random variables $\xi$ defined on a probability space $(\Omega, \Sigma, P)$ that have a finite second moment, i.e., $E\xi^2 < \infty$. The scalar product and norm in $\mathcal{H}$ are given by $(\xi, \eta) = E\xi\eta$ and $\|\eta\| = (\xi, \xi)^{\frac{1}{2}}$, respectively. Let $\xi(t)$ be a real random process such that $\xi(t) \in \mathcal{H}\ \forall t \in [t_0, T]$. For this process, m.s. continuity and m.s. differentiability are defined as the corresponding notions for functions ranging in $\mathcal{H}$ (see [7, Chapter I]).

Let $[t_0, T]$ be an extended segment of the real axis. A continuous random process with fuzzy states (a fuzzy random process, FRP) $\tilde{X}(t)$ is a mapping $\tilde{X} : [t_0, T] \to \mathcal{X}$, i.e., a function $\tilde{X}(t) = \tilde{X}(\omega, t)$ whose values are FRVs from $\mathcal{X}\ \forall t \in [t_0, T]$.

We denote by $X^{\pm}_{\alpha}(\omega, t)$ the $\alpha$-indices of an FRP $\tilde{X}(\omega, t)$. Further considerations will focus on the class of FRPs for which the real functions $X^{\pm}_{\alpha}(\omega, t)$ are jointly square summable (square summable in the aggregate of the variables) on $\Omega \times [0, 1] \times [t_0, T]$.

Let the fuzzy expectation $M(\tilde{X}(t)) = M(\tilde{X}(\omega, t))$ of an FRP $\tilde{X}(\omega, t)\ \forall t \in [t_0, T]$ be defined as the fuzzy expectation (1) of the corresponding FRV with the $\alpha$-indices equal to

$$[M(\tilde{X}(t))]^{\pm}_{\alpha} = EX^{\pm}_{\alpha}(\omega, t), \qquad (\forall\, \alpha \in [0, 1]). \tag{4}$$

The properties of the fuzzy expectations of FRVs (see [5, 18]) imply the following result.

**Proposition 1.** *The fuzzy expectations of FRPs possess the following properties:*

*1. For a nonrandom function* $\tilde{z} : [t_0, T] \to J$, $M(\tilde{z}(t)) = \tilde{z}(t)$.

*2. If* $\varphi : [t_0, T] \to R$ *is a nonrandom scalar factor and* $\tilde{X}(t)$ *is an FRP, then* $M(\varphi(t)\tilde{X}(t)) = \varphi(t)M(\tilde{X}(t))$.

*3. For FRPs* $\tilde{X}(t)$ *and* $\tilde{Y}(t)$, $M(\tilde{X}(t) + \tilde{Y}(t)) = M(\tilde{X}(t)) + M(\tilde{Y}(t))$.

According to (2), the expectation of an FRP $\tilde{X}(t)\ \forall\, t \in [t_0, T]$ is given by

$$m(\tilde{X}(t)) = \frac{1}{2}\int\limits_0^1 \left([M(\tilde{X}(t))]_\alpha^- + [M(\tilde{X}(t))]_\alpha^+\right) d\alpha.$$

*Example 1.* Let real random processes $\xi_i(\omega, t)$ ($i = 1, 2, 3$; $\omega \in \Omega$, $t \in [t_0, T]$) be square summable on $\Omega \times [t_0, T]$ and $\xi_1(\omega, t) < \xi_2(\omega, t) < \xi_3(\omega, t)$ for all $\omega \in \Omega$, $t \in [t_0, T]$.

Consider an FRP $\tilde{X}(t)$ for which the fuzzy number $\tilde{X}(\omega, t)$ has the triangular form ($\xi_1(\omega, t)$, $\xi_2(\omega, t)$, $\xi_3(\omega, t)$) for all $\omega \in \Omega$ and $t \in [t_0, T]$. In other words, the membership function of $\tilde{X}(\omega, t)$ $\forall \omega \in \Omega$, $t \in [t_0, T]$ is described by

$$\mu_{\omega,t}(x) = \begin{cases} \dfrac{x - \xi_1(\omega, t)}{\xi_2(\omega, t) - \xi_1(\omega, t)} & \text{if } x \in [\xi_1(\omega, t), \xi_2(\omega, t)]; \\ \dfrac{x - \xi_3(\omega, t)}{\xi_2(\omega, t) - \xi_3(\omega, t)} & \text{if } x \in [\xi_2(\omega, t), \xi_3(\omega, t)]; \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the $\alpha$-indices of $\tilde{X}(t)$ are

$$X_\alpha^-(t) = (1 - \alpha)\xi_1(t) + \alpha\xi_2(t), \qquad X_\alpha^+(t) = (1 - \alpha)\xi_3(t) + \alpha\xi_2(t). \tag{5}$$

Due to (4) and (5), the fuzzy expectation $M(\tilde{X}(t))$ is given by the following formulas for the $\alpha$-indices:

$$[M(\tilde{X}(t))]_\alpha^- = (1 - \alpha)E\xi_1(t) + \alpha E\xi_2(t)\ (\forall\, \alpha \in [0, 1]),$$
$$[M(\tilde{X}(t))]_\alpha^+ = (1 - \alpha)E\xi_3(t) + \alpha E\xi_2(t)\ (\forall\, \alpha \in [0, 1]).$$

In addition, by (2), the FRP $\tilde{X}(t)$ has the expectation

$$m(\tilde{X}(t)) = \frac{1}{4}(E\xi_1(t) + 2E\xi_2(t) + E\xi_3(t)).$$

We proceed to the notion of the covariance function of an FRP and its properties. In accordance with (3), let the covariance function of an FRP $\tilde{X}(t)$ be defined as

$$K_{\tilde{X}}(t, s) = cov(\tilde{X}(t), \tilde{X}(s)) = \frac{1}{2}\int\limits_0^1 \left(K_{X_\alpha^-}(t, s) + K_{X_\alpha^+}(t, s)\right) d\alpha. \tag{6}$$

Here, $K_{X_\alpha^-}(t, s)$ and $K_{X_\alpha^+}(t, s)$ represent the covariance functions of the real random processes $X_\alpha^-(t)$ and $X_\alpha^+(t)$, respectively, given by

$$K_{X_\alpha^\pm}(t, s) = E\left(X_\alpha^\pm(t) - E(X_\alpha^\pm(t))\right)\left(X_\alpha^\pm(s) - E(X_\alpha^\pm(s))\right).$$

The variance of an FRP $\tilde{X}(t)$ is $D_{\tilde{X}}(t) = K_{\tilde{X}}(t, t)$.

*Example 2.* Within Example 1, let the random processes $\xi_1(t)$ and $\xi_2(t)$, as well as $\xi_2(t)$ and $\xi_3(t)$, be pairwise uncorrelated. Then the covariance function $K_{\tilde{X}}(t_1, t_2)$ of the FRP $\tilde{X}(t)$ of the triangular form ($\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$) is expressed through the covariance functions $K_{\xi_1}(t_1, t_2)$, $K_{\xi_2}(t_1, t_2)$, and $K_{\xi_3}(t_1, t_2)$ of the random processes $\xi_1(t)$, $\xi_2(t)$, and $\xi_3(t)$, respectively, as follows:

$$K_{\tilde{X}}(t_1, t_2) = \frac{1}{6}\left\{K_{\xi_1}(t_1, t_2) + 2K_{\xi_2}(t_1, t_2) + K_{\xi_3}(t_1, t_2)\right\}.$$

Indeed, by formula (5), assuming the uncorrelatedness of $\xi_1(t)$ and $\xi_2(t)$, for the left index $X_\alpha^-(t)$ of the triangular FRP we obtain

$$K_{X_\alpha^-}(t_1,t_2) = (1-\alpha)^2 K_{\xi_1}(t_1,t_2) + \alpha^2 K_{\xi_2}(t_1,t_2).$$

Similarly, based on the uncorrelatedness of $\xi_3(t)$ and $\xi_2(t)$,

$$K_{X_\alpha^+}(t_1,t_2) = (1-\alpha)^2 K_{\xi_3}(t_1,t_2) + \alpha^2 K_{\xi_2}(t_1,t_2).$$

Then definition (6) of the covariance function of an FRP finally leads to the desired conclusion.

According to (6) and the properties of the covariances of real random processes (see [16, Chapter 23]), we arrive at the following result.

**Proposition 2.** *The covariance function of an FRP $\tilde{X}(t)$ possesses the following properties:*

*1. $K_{\tilde{X}}(t_1,t_2) = K_{\tilde{X}}(t_2,t_1)$ $\forall\, t_1,t_2 \in [t_0,T]$ (symmetry).*

*2. Let $\tilde{X}(t)$ be an FRP and $\varphi(t)$ be a nonrandom real function. For an FRP $\tilde{Y}(t) = \varphi(t)\tilde{X}(t)$, we have $K_{\tilde{Y}}(t_1,t_2) = \varphi(t_1)\varphi(t_2)K_{\tilde{X}}(t_1,t_2)$ if $\varphi(t_1)\varphi(t_2) \geqslant 0$.*

*3. If $\tilde{Y}(t) = \tilde{X}(t) + \varphi(t)$, then $K_{\tilde{Y}}(t_1,t_2) = K_{\tilde{X}}(t_1,t_2)$.*

*4. $|K_{\tilde{X}}(t_1,t_2)| \leqslant \sqrt{D_{\tilde{X}}(t_1)D_{\tilde{X}}(t_2)}$.*

An FRP $\tilde{X}(\omega,t)$ with the $\alpha$-intervals $[X_\alpha^-(\omega,t), X_\alpha^+(\omega,t)]$ is said to be continuous at a point $t$ if all its $\alpha$-indices $X_\alpha^\pm(\omega,t)$ are continuous in $t$ as scalar random processes in the mean-square sense [8].

An FRP $\tilde{X}(\omega,t)$ with the $\alpha$-indices $X_\alpha^-(\omega,t)$ and $X_\alpha^+(\omega,t)$ is said to be Seikkala differentiable at a point $t$ if all its $\alpha$-indices are differentiable with respect to $t$ as scalar random processes in the mean-square sense and the derivatives $\frac{\partial}{\partial t}X_\alpha^-(\omega,t)$ and $\frac{\partial}{\partial t}X_\alpha^+(\omega,t)$ are the lower and upper $\alpha$-indices of some FRV, respectively, which is called the derivative at the point $t$ [8]. (Compare this definition with the one for a fuzzy-valued function [19].) In this case, the $t$-derivative of an FRP $\tilde{X}(t)$ will be denoted by $\tilde{X}'(t) = \frac{\partial}{\partial t}\tilde{X}(\omega,t)$.

The second and higher derivatives are sequentially defined in a conventional way.

*Remark 1.* By the definition of an FRP and the arithmetic properties of fuzzy numbers in the interval form, the differentiation of FRPs is a linear operation: the derivative of the sum (difference) of FRPs equals the sum (difference) of the corresponding derivatives, and a constant multiplier can be factored outside the derivative sign.

*Remark 2.* The derivative of an FRV (a constant) coincides with the fuzzy number whose left and right $\alpha$-indices are zero $\forall\,\alpha \in [0,1]$.

The next theorem concerns the fuzzy expectation of the derivative of an FRP.

**Theorem 1.** *Let an FRP $\tilde{X}(t)$ be differentiable in a domain $t \in (t_0,T)$. Then the derivative of the fuzzy expectation of the FRP $\tilde{X}(t)$ is well defined, and the fuzzy expectation of its derivative equals the derivative of its fuzzy expectation:*

$$M\tilde{X}'(t) = (M\tilde{X}(t))'. \tag{7}$$

**Proof.** According to the definitions of the derivative of an FRP and the fuzzy expectation of an FRP, we write

$$[M(\tilde{X}'(t))]_\alpha^\pm = E[\tilde{X}'(t)]_\alpha^\pm = E(X_\alpha^\pm)'(t) = (EX_\alpha^\pm(t))'.$$

The latter equality follows from the corresponding property of real random processes [6, Chapter 6]. Then, using the definition of the Seikkala derivative of the fuzzy-valued function $M\tilde{X}(t)$ [19] and the interval sign of equality of fuzzy numbers, we obtain (7).

Let us emphasize that (7) is the equality of fuzzy-valued functions. Note that Theorem 1 somewhat strengthens the result from [8].

**Corollary 1.** *Under the hypotheses of Theorem 1, the expectations of an FRP and its derivative satisfy the relation*

$$m\tilde{X}'(t) = (m\tilde{X}(t))'. \tag{8}$$

In addition, the following result is valid for the covariance function of the derivative of an FRP.

**Theorem 2** [8]. *Let the covariance functions of the $\alpha$-indices $X_\alpha^\pm(t)$ of an FRP $\tilde{X}(t)$ have well-defined second derivatives $\frac{\partial^2 K_{X_\alpha^-}(t,s)}{\partial t \partial s}$ and $\frac{\partial^2 K_{X_\alpha^+}(t,s)}{\partial t \partial s}$ jointly continuous in the variables $t$, $s$, and $\alpha$. Then the covariance function $K_{\tilde{X}'}(t,s)$ of the derivative $\tilde{X}'(t)$ of the FRP $\tilde{X}(t)$ is given by*

$$K_{\tilde{X}'}(t,s) = \frac{\partial^2 K_{\tilde{X}}(t,s)}{\partial t \partial s}. \tag{9}$$

## 4. STATIONARY FUZZY RANDOM PROCESSES

As is well known, a real random process $\xi(t)$ with $E|\xi(t)|^2 < \infty$, $t \in [0, \infty)$, is said to be stationary (in the wide sense) if it has a constant expectation $E\xi(t) = a$ and the covariance function $E[\xi(t) - a][\xi(s) - a] = K_\xi(t - s)$ that depends only on the difference of the arguments; for details, see [6, Chapter 7; 7, Chapter VII].

We say that an FRP $\tilde{X}(t)$, $t \in [0, \infty)$, is stationary if its $\alpha$-indices $\forall\, \alpha \in [0, 1]$ are real stationary random processes.

*Example 3.* Within Example 2, let all the random processes $\xi_j(t)$ $(j = 1, 2, 3;\ t \in [0, \infty))$ be stationary. Then the triangular FRP $\tilde{X}(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ is stationary.

This fact follows from the expressions for the fuzzy expectations and covariance functions of the triangular FRP $\tilde{X}(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ derived in Examples 1 and 2.

**Theorem 3.** *Let $\tilde{X}(t)$, $t \in [0, \infty)$, be a stationary FRP. Then its fuzzy expectation $M(\tilde{X}(t))$ and expectation $m(\tilde{X}(t))$ are constant, and the covariance function $K_{\tilde{X}}(t_1, t_2) = K_{\tilde{X}}(t_2 - t_1)$ depends on the difference $(t_2 - t_1) = \tau$ of the arguments.*

**Proof.** Assume that $\tilde{X}(t)$ is a stationary FRP. We denote by $m_\alpha^\pm$ the constant expectations of the $\alpha$-indices $X_\alpha^\pm(t)$ of the FRP $\tilde{X}(t)$ $\forall\, \alpha \in [0, 1]$. By definition (3), they are the $\alpha$-indices of the fuzzy expectation $M_\alpha^\pm = m_\alpha^\pm$. Then the fuzzy expectation $M(\tilde{X}(t))$ is constant, and the expectation $m(\tilde{X}(t)) = \frac{1}{2}\int_0^1 (m_\alpha^+ + m_\alpha^-) d\alpha$ is constant as well.

We denote by $K_{X_\alpha^\pm}(t_1, t_2)$ the covariance functions of the $\alpha$-indices, i.e., those of the real random processes $X_\alpha^\pm(t)$. According to the assumption, $X_\alpha^\pm(t)$ is a stationary random process; hence, $K_{X_\alpha^\pm}(t_1, t_2) = K_{X_\alpha^\pm}(t_2 - t_1)$. In this case, by definition (6), the covariance function $K_{\tilde{X}}(t_1, t_2)$ of the FRP $\tilde{X}(t)$ depends on the difference $(t_2 - t_1) = \tau$ of the arguments.

**Theorem 4.** *The covariance function $K_{\tilde{X}}(\tau)$ of a stationary FRP $\tilde{X}(t)$ possesses the following properties:*

*1.* $K_{\tilde{X}}(\tau) = K_{\tilde{X}}(-\tau)$ *(evenness).*

*2. The variance of the stationary FRP $\tilde{X}(t)$ is constant and equals $D_{\tilde{X}} = K_{\tilde{X}}(0)$.*

*3.* $|K_{\tilde{X}}(\tau)| \leqslant K_{\tilde{X}}(0)$ $(\forall\, \tau \in R)$.

Theorem 4 follows from the corresponding properties of the covariance functions $K_{X_\alpha^-}(\tau)$ and $K_{X_\alpha^+}(\tau)$ of the $\alpha$-indices (see [16, Chapter 24]) and the representation (6).

For stationary FRPs, Theorem 2 can be refined as follows.

**Theorem 5.** *Under the conditions of Theorem 2, the covariance function of the derivative* $\tilde{X}'(t)$ *of a differentiable stationary FRP* $\tilde{X}(t)$ *equals the second derivative of its covariance function taken with the minus sign:* $K_{\tilde{X}'}(\tau) = -K''_{\tilde{X}}(\tau)$.

**Proof.** Due to formula (9), $\forall\, t_1, t_2 \in [0, \infty)$ we have

$$K_{\tilde{X}'}(t_1, t_2) = \frac{\partial^2 K_{\tilde{X}}(t_1, t_2)}{\partial t_1 \partial t_2}.$$

By the assumption, $\tilde{X}(t)$ is a stationary FRP. According to Theorem 3, its covariance function depends on the difference of the arguments: $K_{\tilde{X}}(t_1, t_2) = K_{\tilde{X}}(\tau)$, where $\tau = (t_2 - t_1)$. Consequently,

$$K_{\tilde{X}'}(t_1, t_2) = \frac{\partial^2 K_{\tilde{X}}(\tau)}{\partial t_1 \partial t_2} = \frac{\partial}{\partial t_1}\left(\frac{\partial K_{\tilde{X}}(\tau)}{\partial t_2}\right) = \frac{\partial}{\partial t_1}\left(\frac{\partial K_{\tilde{X}}(\tau)}{\partial \tau}\frac{\partial \tau}{\partial t_2}\right)$$
$$= \frac{d^2 K_{\tilde{X}}(\tau)}{d\tau^2}\frac{\partial \tau}{\partial t_1} = K''_{\tilde{X}}(\tau)(-1) = -K''_{\tilde{X}}(\tau).$$

In these formulas, we have utilized the equalities $\frac{\partial \tau}{\partial t_1} = -1$ and $\frac{\partial \tau}{\partial t_2} = 1$. Thus, the covariance function of the FRP $\tilde{X}'(t)$ depends only on the difference of the arguments, and the desired conclusion follows.

**Proposition 3.** *The derivative* $\tilde{X}'(t)$ *of a stationary differentiable FRP* $\tilde{X}(t)$, $t \in [0, \infty)$, *is a stationary FRP.*

Indeed, by the condition, the $\alpha$-indices $X^{\pm}_{\alpha}(t)$, $t \in [0, \infty)$, of the FRP $\tilde{X}(t)$ are real stationary random processes $\forall\, \alpha \in [0, 1]$. Then, based on the well-known property of real stationary processes [6, Chapter 7], their derivatives $(X^{\pm}_{\alpha})'(t)$ are such as well. Therefore, Proposition 3 is immediate from the definition of the differentiability of FRPs.

**Proposition 4.** *Let* $\tilde{Y}(t)$, $t \in [0, \infty)$, *be a stationary FRP* $k$ *times differentiable on* $(0, \infty)$ *and at least one of given constants* $b_s \geqslant 0$ $(s = 0, 1, \ldots, k)$ *be nonzero. Then the linear combination of the derivatives,* $\tilde{Z}(t) = \sum_{s=0}^{k} b_s \tilde{Y}^{(s)}(t)$, *is a stationary FRP.*

Indeed, under the hypotheses of Proposition 4, the left and right indices $Z^{\pm}_{\alpha}(t)$ of the FRP $\tilde{Z}(t)$ $\forall\, \alpha \in [0, 1]$ have the form

$$Z^{\pm}_{\alpha}(t) = \left[\sum_{s=0}^{k} b_s (\tilde{Y})^{(s)}(t)\right]^{\pm}_{\alpha} = \sum_{s=0}^{k} b_s (Y^{\pm}_{\alpha}(t))^{(s)}.$$

Here, we have utilized the definition of the derivatives of FRPs and the properties of the arithmetical operations over fuzzy numbers in the interval form.

Since $Y^{\pm}_{\alpha}(t)$ are stationary real processes, by the well-known result for such processes [6, Chapter 7], $Z^{\pm}_{\alpha}(t)$ are stationary real random processes, and Proposition 4 is proved accordingly.

## 5. THE SPECTRAL DENSITY OF A STATIONARY FRP. THE GENERALIZED WIENER–KHINCHIN THEOREM

Consider the spectral representation problem of the covariance function of a stationary FRP.

The following result is well-known for a real stationary random process $\xi(t)$ defined on an infinite time interval $[0, \infty)$; for example, see [6, Chapter 7; 7, Chapter VII].

**Lemma 1** (the Wiener–Khinchin theorem). *The covariance function* $K_{\xi}(\tau)$ *and spectral density* $S_{\xi}(\tau)$ *of a real stationary random process* $\xi(t)$ *are related by the self-reciprocal inverse Fourier* *cosine transforms:*

$$K_\xi(\tau) = \int\limits_0^\infty S_\xi(\omega)\cos\omega\tau d\omega, \quad S_\xi(\omega) = \frac{2}{\pi}\int\limits_0^\infty K_\xi(\tau)\cos\omega\tau d\tau. \tag{10}$$

*Remark 3* [7, Chapter VII. ] The existence of the spectral density $S_\xi(\omega)$ of a real stationary random process $\xi(t)$ and the relations (10) are ensured, e.g., by the continuity of the covariance function $K_\xi(\tau)$ of the process $\xi(t)$ and its summability on $(0,\infty)$ (i.e., $\int_0^\infty |K_\xi(\tau)|d\tau < \infty$).

Now we generalize Lemma 1 to the case of stationary FRPs. Let a stationary FRP $\tilde{X}(t)$ defined on $[0,\infty)$ have the $\alpha$-indices $X_\alpha^\pm(t)$, and let $S_{X_\alpha^\pm}(\omega)$ be the spectral densities of the stationary random processes $X_\alpha^\pm(t)$ $(\forall\,\alpha \in [0,1])$ such that the functions $S_{X_\alpha^\pm}(\omega)$ are summable in $\alpha$ on $[0,1]$. We call the function

$$S_{\tilde{X}}(\omega) = \frac{1}{2}\int\limits_0^1 \left(S_{X_\alpha^+}(\omega) + S_{X_\alpha^-}(\omega)\right) d\alpha \tag{11}$$

the spectral density of a stationary FRP $\tilde{X}(t)$.

*Example 4.* Within Example 3, we denote by $S_{\xi_i}(\omega)$ the spectral densities of the real stationary random processes $\xi_i(t)$ $(i = 1,2,3)$. Then the spectral density of the stationary FRP $\tilde{X}(t)$ of the triangular form $(\xi_1(t),\xi_2(t),\xi_3(t))$ is given by

$$S_{\tilde{X}}(\omega) = \frac{1}{6}\left(S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)\right).$$

Indeed, let us denote by $K_{\xi_i}(\tau)$ the covariance functions of the real random processes $\xi_i(t)$. Due to (5) and the pairwise uncorrelatedness of the random processes $\xi_1(t)$ and $\xi_2(t)$, as well as of $\xi_2(t)$ and $\xi_3(t)$, for the covariance functions of the $\alpha$-indices $X_\alpha^\pm(t)$ of the FRP $\tilde{X}(t)$ we write

$$K_{X_\alpha^-}(\tau) = (1-\alpha)^2 K_{\xi_1}(\tau) + \alpha^2 K_{\xi_2}(\tau),$$
$$K_{X_\alpha^+}(\tau) = (1-\alpha)^2 K_{\xi_3}(\tau) + \alpha^2 K_{\xi_2}(\tau).$$

Then, based on formulas (10) for the spectral densities of the real stationary random processes $X_\alpha^\pm$, it follows that

$$S_{X_\alpha^-}(\omega) = (1-\alpha)^2 S_{\xi_1}(\omega) + \alpha^2 S_{\xi_2}(\omega),$$
$$S_{X_\alpha^+}(\omega) = (1-\alpha)^2 S_{\xi_3}(\omega) + \alpha^2 S_{\xi_2}(\omega).$$

Therefore, according to (11), the spectral density of the FRP $\tilde{X}(t)$ of the triangular form $(\xi_1(t),\xi_2(t),\xi_3(t))$ is given by

$$S_{\tilde{X}}(\omega) = \frac{1}{2}\left(\int\limits_0^1(1-\alpha)^2 d\alpha S_{\xi_1}(\omega) + \int\limits_0^1 \alpha^2 d\alpha S_{\xi_2}(\omega) + \int\limits_0^1(1-\alpha)^2 d\alpha S_{\xi_3}(\omega) + \int\limits_0^1 \alpha^2 d\alpha S_{\xi_2}(\omega)\right)$$
$$= \frac{1}{6}\left(S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)\right),$$

which finally implies Proposition 4.

The next result is valid by definition (11) and the properties of the spectral densities of real stationary random processes (see [16, Chapter 24]).

**Proposition 5.** *The spectral density of a stationary FRP possesses the following properties:*

*1. The spectral density of a stationary FRP $\tilde{X}(t)$ is nonnegative, i.e., $S_{\tilde{X}}(\omega) \geqslant 0$.*

*2. Integrating the spectral density of a stationary FRP $\tilde{X}(t)$ over $\omega$ between zero and infinity gives the variance of the FRP $\tilde{X}(t)$, i.e., $\int_0^\infty S_{\tilde{X}}(\omega)d\omega = D_{\tilde{X}}$.*

The appropriateness of the above definition (11) is confirmed by the generalized Wiener–Khinchin theorem established below.

**Theorem 6.** *Let $\tilde{X}(t)$, $t \in [0, \infty)$, be a stationary FRP such that the covariance function $K_{X_\alpha^\pm}(\tau)$ and spectral density $S_{X_\alpha^\pm}(\omega)$ are well-defined for its $\alpha$-indices $X_\alpha^\pm(t)$ for any $\alpha \in [0, 1]$ and, moreover, are jointly summable on $[0, \infty) \times [0, 1]$. Then the covariance function and spectral density of the stationary FRP $\tilde{X}(t)$ are related by the self-reciprocal inverse Fourier cosine transforms:*

$$K_{\tilde{X}}(\tau) = \int_0^\infty S_{\tilde{X}}(\omega) \cos \omega\tau d\omega, \qquad S_{\tilde{X}}(\omega) = \frac{2}{\pi} \int_0^\infty K_{\tilde{X}}(\tau) \cos \omega\tau d\tau. \tag{12}$$

**Proof.** We begin with deriving the first formula in (12). For stationary real processes $X_\alpha^\pm(t)$, by Lemma 1, $K_{X_\alpha^-}(\tau) = \int_0^\infty S_{X_\alpha^-}(\omega) \cos \omega\tau d\omega$ and $K_{X_\alpha^+}(\tau) = \int_0^\infty S_{X_\alpha^+}(\omega) \cos \omega\tau d\omega$. Summing both sides of these equalities and integrating the resulting expression over $\alpha$ between 0 and 1 yield $\int_0^1 (K_{X_\alpha^-}(\tau) + K_{X_\alpha^+}(\tau))d\alpha = \int_0^1 \int_0^\infty (S_{X_\alpha^-}(\omega) + S_{X_\alpha^+}(\omega)) \cos \omega\tau d\omega d\alpha$. Changing the order of integration on the right-hand side (based on Fubini's theorem) and using (6) and (11), we finally arrive at the desired formula for $K_{\tilde{X}}(\tau)$.

In addition, according to Lemma 1, $S_{X_\alpha^\pm}(\omega) = \frac{2}{\pi} \int_0^\infty K_{X_\alpha^\pm}(\tau) \cos \omega\tau d\tau$. In view of (6) and (11), similar considerations as above lead to the second formula in (12).

*Example 5.* Within Example 4, by Theorem 6, the covariance function $K_{\tilde{X}}(t)$ of the triangular stationary fuzzy random process $\tilde{X}(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ is given by

$$K_{\tilde{X}}(\tau) = \frac{1}{6} \int_0^\infty (S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)) \cos \omega\tau d\omega = \frac{1}{6} \left\{K_{\xi_1}(\tau) + 2K_{\xi_2}(\tau) + K_{\xi_3}(\tau)\right\},$$

where $K_{\xi_i}(\tau)$ $(i = 1, 2, 3)$ denote the covariance functions of the real stationary random processes $\xi_i(t)$.

Note that this result agrees with Example 2.

## 6. ON THE TRANSFORMATION OF A STATIONARY FRP BY A LINEAR DYNAMIC TIME-INVARIANT SYSTEM

Suppose that a random signal $\xi(t)$ is supplied to the input of some device, and a random signal $\eta(t)$ is accordingly observed at its output. The device is called a linear dynamic time-invariant system of the $n$th order if the output $\eta(t)$ and input $\xi(t)$ are related by a linear differential equation of the $n$th order with constant coefficients of the form

$$\begin{aligned} a_n\eta^{(n)}(t) + a_{n-1}\eta^{(n-1)}(t) + \cdots + a_1\eta'(t) + a_0\eta(t) \\ = b_k\xi^{(k)}(t) + b_{k-1}\xi^{(k-1)}(t) + \cdots + b_1\xi'(t) + b_0\xi(t) \quad (t > 0). \end{aligned} \tag{13}$$

Here, $a_j$ $(j = 0, 1, \ldots, n)$ and $b_s$ $(s = 0, 1, \ldots, k)$ are real numbers, and $k < n$.

If the dynamic system (13) is asymptotically stable and a real stationary random process $\xi(t)$ is supplied to its input, then the output random process $\eta(t)$ can be considered stationary for sufficiently large values of $t$, i.e., after some transient period. The problem of establishing the relationship between the numerical characteristics (expectations and covariance functions) of the input and output real stationary random signals of the dynamical system (13) is widely known in the literature; for example, see [6, Chapter 7].

Suppose that a stationary FRP $\tilde{Y}(t)$ is supplied to the input of the dynamic system (13) and a stationary FRP $\tilde{X}(t)$ is accordingly observed at its output.

It is required to calculate the characteristics of the output stationary FRP $\tilde{X}(t)$ from the known characteristics of the input stationary FRP $\tilde{Y}(t)$ of the dynamic system (13).

Consider the problem of calculating the constant fuzzy expectation $M\tilde{X}$ (or the expectation $m\tilde{X}$) at the output of system (13) from the known constant fuzzy expectation $M\tilde{Y}$ (or the expectation $m\tilde{Y}$) at its input.

**Proposition 6.** *Let a stationary $k$ times differentiable FRP $\tilde{Y}(t)$, $t \in [0,\infty)$, be supplied to the input of the dynamic system* (13) *and a stationary $n$ times continuously differentiable FRP $\tilde{X}(t)$, $t \in [0,\infty)$, be accordingly observed at its output. Then*

$$M\tilde{X} = \frac{b_0}{a_0} M\tilde{Y}, \qquad m\tilde{X} = \frac{b_0}{a_0} m\tilde{Y}.$$

Indeed, by the condition,

$$\begin{aligned} &a_n\tilde{X}^{(n)}(t) + a_{n-1}\tilde{X}^{(n-1)}(t) + \cdots + a_1\tilde{X}'(t) + a_0\tilde{X}(t) \\ &= b_k\tilde{Y}^{(k)}(t) + b_{k-1}\tilde{Y}^{(k-1)}(t) + \cdots + b_1\tilde{Y}'(t) + b_0\tilde{Y}(t)\ (t > 0). \end{aligned} \tag{14}$$

We equate the fuzzy expectations of the left- and right-hand sides of (14). Based on the algebraic properties of fuzzy expectations (Proposition 1) and the properties of the derivative of fuzzy expectations (Theorem 1), it follows that

$$\begin{aligned} &a_n(M\tilde{X})^{(n)}(t) + a_{n-1}(M\tilde{X})^{(n-1)}(t) + \cdots + a_1(M\tilde{X})'(t) + a_0M\tilde{X}(t) \\ &= b_k(M\tilde{Y})^{(k)}(t) + b_{k-1}(M\tilde{Y})^{(k-1)}(t) + \cdots + b_1(M\tilde{Y})'(t) + b_0M\tilde{Y}(t). \end{aligned} \tag{15}$$

Since the fuzzy expectations of the stationary FRPs $\tilde{Y}(t)$ and $\tilde{X}(t)$ are constant, their derivatives of any order equal a fuzzy number whose left and right indices are all zero (see Remark 2). Then (15) entails the equality

$$a_0M\tilde{X} = b_0M\tilde{Y},$$

and, consequently, $M\tilde{X} = \frac{b_0}{a_0}M\tilde{Y}$. By analogy, we establish $m\tilde{X} = \frac{b_0}{a_0}m\tilde{Y}$ for their expectations.

Next, using the known covariance function $K_{\tilde{Y}}(\tau)$ of the input stationary FRP $\tilde{Y}(t)$ of the dynamic system (14), we find the covariance function $K_{\tilde{X}}(\tau)$ and variance $D_{\tilde{X}}$ of its output stationary FRP $\tilde{X}(t)$.

**Theorem 7.** *Let the coefficients of the dynamic system* (14) *be nonnegative, i.e., $a_j \geqslant 0$ ($j = 0,1,\ldots,n$), $b_s \geqslant 0$ ($s = 0,1,\ldots,k$). Let a $k$ times continuously differentiable FRP $\tilde{Y}(t)$, $t \in [0,\infty)$, be supplied to the input of the dynamic system* (14), *and let the covariance functions $K_{Y_\alpha^\pm}(\tau)$ and spectral densities $S_{Y_\alpha^\pm}(\omega)$ be well defined for its $\alpha$-indices $Y_\alpha^\pm(t)$ for any $\alpha \in [0,1]$ and, moreover, be jointly summable on $[0,\infty) \times [0,1]$.*

*Let an $n$ times continuously differentiable FRP $\tilde{X}(t)$, $t \in [0,\infty)$, be accordingly observed at the output of the dynamic system* (14). *Then the algorithm for calculating (in real form) the covariance function $K_{\tilde{X}}(\tau)$ of the output stationary FRP $\tilde{X}(t)$ includes the following stages:*

*1) Find the spectral density $S_{\tilde{Y}}(\omega)$ of the input stationary FRP $\tilde{Y}(t)$ from its covariance function $K_{\tilde{Y}}(\tau)$ using the generalized Wiener–Khinchin formula* (12):

$$S_{\tilde{Y}}(\omega) = \frac{2}{\pi}\int\limits_0^{\infty} K_{\tilde{Y}}(\tau)\cos\omega\tau d\tau. \tag{16}$$

*2) Find the frequency response $\Phi(i\omega)$ for the differential equation* (14):

$$\Phi(i\omega) = \frac{b_k(i\omega)^k + \cdots + b_1(i\omega) + b_0}{a_n(i\omega)^n + \cdots + a_1(i\omega) + a_0}, \tag{17}$$

*where $i$ indicates the imaginary unit.*

*3) Find the spectral density $S_{\tilde{X}}(\omega)$ of the output stationary FRP $\tilde{X}(t)$ from the spectral density $S_{\tilde{Y}}(\omega)$ of the input stationary FRP $\tilde{Y}(t)$ and the squared absolute value $|\Phi(i\omega)|^2$ of the frequency response* (17):

$$S_{\tilde{X}}(\omega) = |\Phi(i\omega)|^2 S_{\tilde{Y}}(\omega). \tag{18}$$

*4) Find the covariance function $K_{\tilde{X}}(\tau)$ and (or) variance $D_{\tilde{X}}$ of the output stationary FRP $\tilde{X}(t)$ from its spectral density $S_{\tilde{X}}(\omega)$* (18) *using the generalized Wiener–Khinchine formula* (12):

$$K_{\tilde{X}}(\tau) = \int\limits_0^{\infty} S_{\tilde{X}}(\omega)\cos\omega\tau d\omega, \qquad D_{\tilde{X}} = \int\limits_0^{\infty} S_{\tilde{X}}(\omega)d\omega. \tag{19}$$

**Proof.** Due to equation (14), the definition of derivatives for FRPs, the assumption on the positive coefficients $a_k$ and $b_s$, and the definition of summation for fuzzy numbers in the interval form, for the $\alpha$-indices $X_\alpha^{\pm}(t)$ and $Y_\alpha^{\pm}(t)$ of the FRPs $\tilde{X}(t)$ and $\tilde{Y}(t)$, respectively, we have

$$\begin{aligned} &a_n(X_\alpha^{\pm})^{(n)}(t) + a_{n-1}(X_\alpha^{\pm})^{(n-1)}(t) + \cdots + a_1(X_\alpha^{\pm})'(t) + a_0X_\alpha^{\pm}(t) \\ &= b_k(Y_\alpha^{\pm})^{(k)}(t) + b_{k-1}(Y_\alpha^{\pm})^{(k-1)}(t) + \cdots + b_1(Y_\alpha^{\pm})'(t) + b_0Y_\alpha^{\pm}(t) \quad (t > 0). \end{aligned} \tag{20}$$

By the hypothesis of this theorem, the $\alpha$-indices $Y_\alpha^{\pm}(t)$ and $X_\alpha^{\pm}(t)$ are real stationary random processes for $t \in [0,\infty)$. For each pair of real stationary processes $Y_\alpha^{\pm}(t)$ and $X_\alpha^{\pm}(t)$ related by the dynamic system (20), we apply the well-known algorithm consisting of Stages 1)–4) (see [6, Chapter 7]).

First, for any $\alpha \in [0,1]$, we find the spectral density $S_{Y_\alpha^{\pm}}(\omega)$ of the input random process $Y_\alpha^{\pm}(t)$ of the dynamic system (20) from its covariance function $K_{Y_\alpha^{\pm}}(\tau)$ using (10):

$$S_{Y_\alpha^{\pm}}(\omega) = \frac{2}{\pi}\int\limits_0^{\infty} K_{Y_\alpha^{\pm}}(\tau)\cos\omega\tau d\tau.$$

Next, using the frequency response $\Phi(i\omega)$, we calculate the spectral densities of the output stationary real random processes $X_\alpha^{\pm}(t)$ by the formulas

$$S_{X_\alpha^{\pm}}(\omega) = |\Phi(i\omega)|^2 S_{Y_\alpha^{\pm}}(\omega).$$

In view of (11), this result implies (18).

We proceed with determining the covariance functions

$$K_{X_\alpha^\pm}(\tau) = \int_0^\infty S_{X_\alpha^\pm}(\omega)\cos\omega\tau d\omega$$

of the output real stationary random processes $X_\alpha^\pm(t)$ of the dynamic system (20).

After that, we apply the above definitions of the covariance function (6) and spectral density (11) of the stationary FRPs:

$$K_{\tilde X}(\tau) = \frac{1}{2}\int_0^1 (K_{X_\alpha^-}(\tau) + K_{X_\alpha^+}(\tau))d\alpha = \frac{1}{2}\int_0^1 \left(\int_0^\infty (S_{X_\alpha^+}(\omega) + S_{X_\alpha^-}(\omega))\cos\omega\tau d\omega\right) d\alpha.$$

Changing the order of integration based on Fubini's theorem gives

$$K_{\tilde X}(\tau) = \int_0^\infty \left(\frac{1}{2}\int_0^1 (S_{X_\alpha^+}(\omega) + S_{X_\alpha^-}(\omega))d\alpha\right)\cos\omega\tau d\omega = \int_0^\infty S_{\tilde X}(\omega)\cos\omega\tau d\omega.$$

Thus, the first formula in (19) has been derived. The formula for the variance $D_{\tilde X}$ follows from the former one since $D_{\tilde X} = K_{\tilde X}(0)$.

*Remark 4.* Equation (20) can be interpreted as an equation in the Hilbert space $\mathcal{H}$ of random variables with a finite second moment. If the real parts of all roots of the characteristic equation $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$ are negative, then equation (20) is asymptotically Lyapunov stable in the space $\mathcal{H}$ (see [21, Chapter II]).

*Remark 5.* Equation (15) is essentially a fuzzy differential equation. Such equations were considered in [19, 22]. Equation (14) is a fuzzy random equation; for details, see [23–25].

*Example 6.* Let an FRP of the triangular form $\tilde Y(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ be supplied to the input of the linear dynamic system (13), where the real random processes $\xi_i(t)$ $(i = 1, 2, 3)$ satisfy the conditions of Example 4. Then, according to Example 3, the FRP $\tilde Y(t)$ is stationary. Due to Example 2, its covariance function has the form

$$K_{\tilde Y}(\tau) = \frac{1}{6}\{K_{\xi_1}(\tau) + 2K_{\xi_2}(\tau) + K_{\xi_3}(\tau)\},$$

where $K_{\xi_i}(\tau)$ $(i = 1, 2, 3)$ are the covariance functions of the real random processes $\xi_i(t)$.

Moreover, according to Example 4, the spectral density of the input stationary triangular FRP $\tilde Y(t) = (\xi_1(t), \xi_2(t), \xi_3(t))S_{\tilde Y}(\omega)$ of the dynamic system (13) is given by

$$S_{\tilde Y}(\omega) = \frac{1}{6}(S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)),$$

where $S_{\xi_i}(\omega)$ $(i = 1, 2, 3)$ are the spectral densities of the real random processes $\xi_i(t)$.

Finally, using Theorem 7, we find the spectral density of the output stationary FRP $\tilde X(t)$ of the dynamic system (13):

$$S_{\tilde X}(\omega) = |\Phi(i\omega)|^2 S_{\tilde Y}(\omega) = \frac{1}{6}|\Phi(i\omega)|^2\{S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)\},$$

where $\Phi(i\omega)$ is the frequency response of system (13). By Theorem 7, the covariance function $K_{\tilde X}(\tau)$ of the output FRP $\tilde X(t)$ of the dynamic system has the form

$$K_{\tilde X}(\tau) = \frac{1}{6}\int_0^\infty |\Phi(i\omega)|^2(S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega))\cos\omega\tau d\omega.$$

## 7. CONCLUSIONS

Theorems 3–7, as well as Propositions 3–6, form the essential content and scientific contribution of this paper. We emphasize the significance of the concept of the spectral density of a stationary FRP (Theorem 6) and the algorithm for calculating the covariance function of a stationary FRP at the output of a dynamic system from the covariance function of a stationary FRP at its input (Theorem 7). Examples 1–6 have illustrated the applicability of the novel theory to triangular FRPs.

The results of this work can be developed in the following directions:

1. They will remain valid if the covariance of fuzzy random variables from [5] is used instead of definition (3).

2. As is known [7], the Wiener–Khinchin theorem for real random processes (Lemma 1) can be written in a more general form: for this purpose, the spectral distribution function and the Riemann–Stiltjes integral should be considered instead of the spectral density and the Riemann integral.

The generalized Wiener–Khinchin theorem (Theorem 6) can be developed in this direction for stationary fuzzy random processes.

3. It is possible to extend some results of this paper to the case of generalized fuzzy numbers (see [26]).

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